Mon Apr 16, 2012

SCORE: 16 / 20 POINTS

NO CALCULATORS ALLOWED
SHOW PROPER WORK & SIMPLIFY ALL ANSWERS
(ANSWERS WITHOUT SOLUTIONS WILL *NOT* EARN FULL CREDIT)

Prove the derivative of $\tanh^{-1} x$ without using the logarithmic definition of $\tanh^{-1} x$. **SCORE: 4 / 4 PTS**
You may use the derivatives of the non-inverse hyperbolic functions that were listed in your textbook without proving them.
NOTE: You must prove the Pythagorean–like identity for $\tanh x$ if you wish to use it.

$y = \tanh^{-1} x$

$\tanh y = x$ $\frac{1}{2}$

$\frac{d}{dx} \tanh y = \frac{d}{dx} x$

$\operatorname{sech}^2 y \frac{dy}{dx} = 1$ 1

$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}$ $\frac{1}{2}$ $= \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}$ $\frac{1}{2}$

$\frac{d}{dx} \tanh^{-1} x = \boxed{\frac{1}{1-x^2}}$

$1 - \tanh^2 y = \operatorname{sech}^2 y \rightarrow$ prove

$1 - \tanh^2 y$

$= 1 - \frac{\sinh^2 y}{\cosh^2 y}$

$= \frac{\cosh^2 y - \sinh^2 y}{\cosh^2 y}$ 1

$= \frac{1}{\cosh^2 y}$

$= \operatorname{sech}^2 y$ $\frac{1}{2}$

Find $\frac{d}{dx} \tanh^{-1}(\operatorname{sech} x)$. **SCORE: 3 / 3 PTS**
You may use the hyperbolic identities and derivatives from your textbook without proving them.

$\frac{d}{dx} \tanh^{-1}(\operatorname{sech} x)$

$1 - \tanh^2 x = \operatorname{sech}^2 x$
$1 - \operatorname{sech}^2 x = \tanh^2 x$

$= \frac{1}{1-\operatorname{sech}^2 x}$ 1 $\cdot -\operatorname{sech} x \tanh x$ 1

$\frac{1}{2}$

$= \frac{-\operatorname{sech} x \cancel{\tanh x}}{\cancel{\tanh^2 x}} = \boxed{\frac{-\operatorname{sech} x}{\tanh x}} = \frac{-1}{\cancel{\cosh x}} \cdot \frac{\cancel{\cosh x}}{\sinh x} = -\operatorname{csch} x$ $\frac{1}{2}$

$\frac{d}{dx} \tanh^{-1}(\operatorname{sech} x) = \boxed{-\operatorname{csch} x}$

Find $\lim_{x \to 0^-} \operatorname{csch} x$. **Do NOT use a graph. Give *BRIEF* algebraic or numerical reasoning.** **SCORE: 2 / 2 PTS**

$\lim_{x \to 0^-} \operatorname{csch} x = \lim_{x \to 0^-} \frac{1}{\sinh x} = \lim_{x \to 0^-} \frac{2}{e^x - e^{-x}}$

$= \boxed{-\infty}$ 1

$\frac{2}{1-1} = \frac{2}{0} = \infty$ ambiguous, can't flip

$\frac{2}{1^- - \frac{1}{1^-}}$

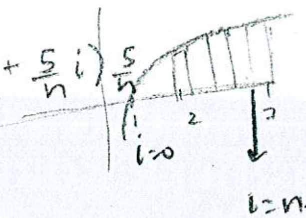
$1^- < \frac{1}{1^-}$ 1 $\frac{2}{0^-} = -\infty$

Write an algebraic expression to approximate the area under $f(x) = \ln x$ over the interval $[2, 7]$ using a left hand sum and n subintervals (as shown in class). **Do NOT evaluate the expression. Your answer must not use $f(\,)$ notation.** SCORE: 3 / 3 PTS

$\Delta x = \frac{b-a}{n} = \frac{7-2}{n} = \frac{5}{n}$

$L_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a + i\Delta x)\Delta x$ $\quad a = 2$ $\quad \Delta x = \frac{5}{n}$

$f(a + \frac{5}{n} i)\frac{5}{n}$



$$\boxed{L_n = \lim_{n\to\infty} \frac{5}{n} \sum_{i=0}^{n-1} \ln\left(2 + \frac{5}{n} i\right)}$$

1 1 1

If $\coth x = -3$, find $\operatorname{sech} x$ and $\sinh x$. SCORE: 1 / 5 PTS

$\frac{1}{\tanh x} = 3$

$\frac{\cosh x}{\sinh x} = -3$

$(\cosh x)^2 = (-3\sinh x)^2$

$\cosh^2 x = 9\sinh^2 x$

$\cosh^2 x = 9(\cosh^2 x - 1)$

$\cosh^2 x = 9\cosh^2 x - 9$

$8\cosh^2 x = 9$

$\cosh^2 x = \frac{9}{8}$

$\cosh x = \pm\sqrt{\frac{9}{8}} = \pm\frac{3}{2\sqrt{2}}$

$\cosh x > 0 \rightarrow \cosh x = \frac{3}{2\sqrt{2}}$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x = 1 + \sinh^2 x$

$$\boxed{\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2\sqrt{2}}{3}}$$ 1

$\sinh x = \pm\sqrt{\cosh^2 x - 1}$

$= \pm\sqrt{\frac{9}{8} - 1}$

$= \pm\sqrt{1/8}$

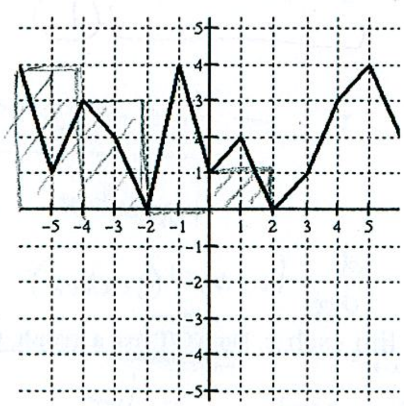
$= \pm \frac{1}{2\sqrt{2}}$

$\sinh x = \frac{1}{2\sqrt{2}} =$

$$\boxed{\sinh x \quad \frac{\sqrt{2}}{4}}$$

$\star \coth x = \frac{\cosh x}{\sinh x} = -3$ $\quad$ cosh x is positive, sinh x is negative

Estimate the area under the function shown on the right over the interval $[-6, 2]$ using the left hand sum with 4 equal width subintervals. SCORE: 3 / 3 PTS



$\Delta x = \frac{b-a}{n} = \frac{2-(-6)}{4} = 2$ $\frac{1}{2}$

$L_4 = \lim_{n\to 4} \sum_{i=0}^{n-1} f(a + i\Delta x)\Delta x$

$L_4 = f(-6)\Delta x + f(-4)\Delta x + f(-2)\Delta x + f(0)\Delta x$

$= 4(2) + 3(2) + 0(2) + 1(2)$ $\quad \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$

$= 8 + 6 + 0 + 2$

$= \boxed{16}$ $\frac{1}{2}$